AD-A044 571 POLYTECHNIC INST OF NEW YORK FARMINGDALE DEPT OF MEC--ETC F/G 20/4
AN INTEGRAL SPLINE METHOD FOR BOUNDARY LAYER EQUATIONS.(U)
JUL 77 S G RUBIN, P K KHOSLA AFOSR-74-2635
UNCLASSIFIED POLY-M/AE-77-12 AFOSR-TR-77-1210 NL

1 OF 1
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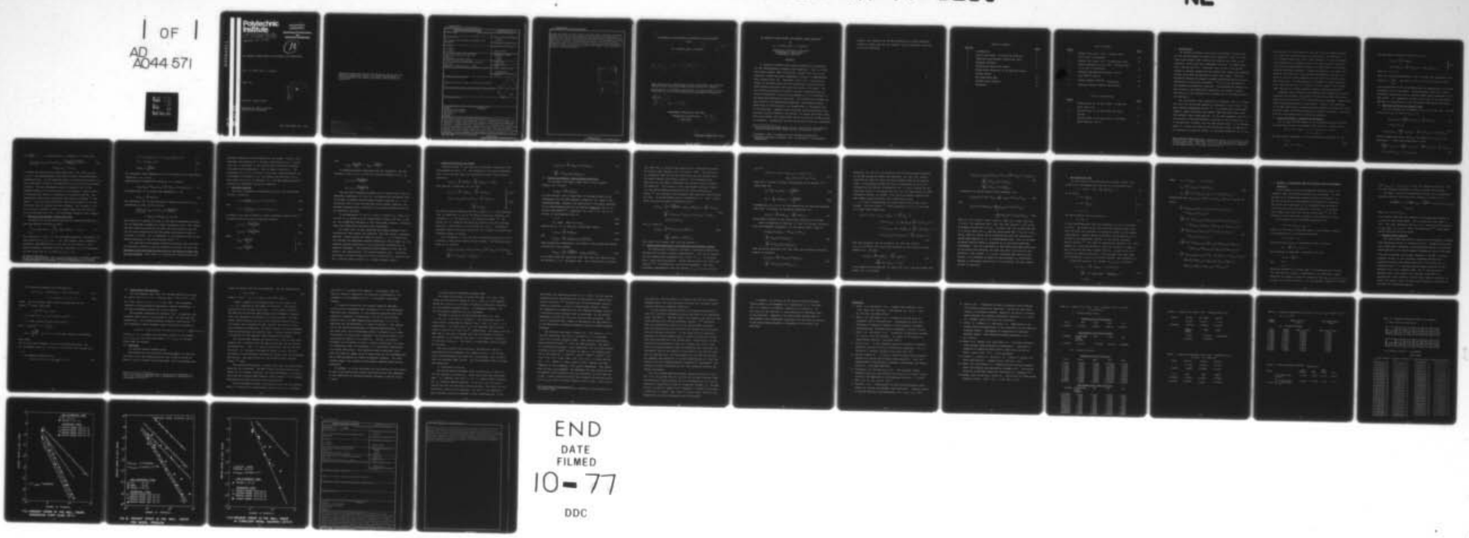

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AN INTEGRAL SPLINE METHOD FOR BOUNDARY LAYER EQUATIONS

By S. G. Rubin and P. K. Khosla

JULY 1977

Grant No. AFOSR 74-2635

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POLY M/AE Report No. 77-12

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER

2. GOVT ACCESSION NO.

3. RECIPIENT'S CATALOG NUMBER

4. TITLE (and Subtitle)
AN INTEGRAL SPLINE METHOD FOR BOUNDARY LAYER EQUATIONS

5. TYPE OF REPORT & PERIOD COVERED
INTERIM

6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s)
S G RUBIN
P K KHOSLA

8. CONTRACT OR GRANT NUMBER(s)
AFOSR 74-2635

9. PERFORMING ORGANIZATION NAME AND ADDRESS
POLYTECHNIC INSTITUTE OF NEW YORK
ROUTE 110
FARMINGDALE, NEW YORK 11735

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
2307A1
61102F

11. CONTROLLING OFFICE NAME AND ADDRESS
AIR FORCE OFFICE OF SCIENTIFIC RESEARCH/NA
BLDG 410
BOLLING AIR FORCE BASE, D C 20332

12. REPORT DATE
July 1977

13. NUMBER OF PAGES
41

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)

15. SECURITY CLASS. (of this report)
UNCLASSIFIED

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Re

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)
SPLINES
POLYNOMIAL INTERPOLATION
MODIFIED FINITE ELEMENT
BOX METHOD
LAMINAR
TURBULENT

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

An integral procedure using spline polynomials is described for the two-dimensional boundary layer equations. This is a modified finite-element (MFE) formulation, wherein each term in the equations, rather than each independent variable, is approximated with a spline curve fit. Therefore, this is not a true finite-element or Galerkin method and the conventional spline relationships between functional and derivative values still apply. The only difference between the present integral formulation and our earlier differential collocation procedures is in the treatment of the governing differential equations. The

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S. G. Rubin ~~and~~ P. K. Khosla

16 2307, 9781   17 A1, 01

This research was supported by the Air Force Office of Scientific Research under Grant No. AFOSR-74-2635, Project No. 9781-01.

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18 AFOSR

19 TR-77-1210

POLYTECHNIC INSTITUTE OF NEW YORK

Aerodynamics Laboratories

12 45p.

11 July 1977

14

POLY-M/AE Report No. 77-12

410 338   1B

AN INTEGRAL SPLINE METHOD FOR BOUNDARY LAYER EQUATIONS[†]

by

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ABSTRACT

An integral procedure using spline polynomials is described for the two-dimensional boundary layer equations. This is a modified finite-element (MFE) formulation, wherein each term in the equations, rather than each independent variable, is approximated with a spline curve fit. Therefore, this is not a true finite-element or Galerkin method and the conventional spline relationships between functional and derivative values still apply. The only difference between the present integral formulation and our earlier differential collocation procedures is in the treatment of the governing differential equations. The differential methods are more suited to non-conservation equations; the present integral formulation is more desirable for conservation or divergence form of the equations. Boundary layer solutions using conventional second-order finite-difference collocation, the second and fourth order Keller Box Scheme, and fourth-order spline collocation or MFE methods are compared. Conservation and non-conservation forms are considered.

[†] This research was supported by the Air Force Office of Scientific Research under Grant No. AFOSR 74-2635, Project No. 9781-01.

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Finally, the extension of the MFE formulation to three-coordinate parabolic systems and for the transonic small disturbance equations is briefly described.

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1. Introduction

An integral procedure using spline polynomials is described for the two dimensional boundary layer equations. This is a modified finite-element (MFE) formulation, wherein each term in the equations, rather than each independent variable, is approximated with a spline curve fit. Therefore, this is not a true finite-element or Galerkin method and the conventional spline relationships between functional and derivation values still apply. The only difference between the present integral formulation and our earlier differential collocation procedures [1-3] is in the treatment of the governing differential equations. The differential methods are more suited to non-conservation equations; the present integral formulation is more desirable for conservation or divergence form of the equations.

For the boundary layer equations in divergence form it is shown here that the spline MFE approach is equivalent to the class of two-point methods proposed by Keller [4]. His second-order development commonly termed the Keller Box Scheme* (KBS) has been widely used for boundary layer investigations. It has been suggested that due to the two-point centered-difference character of this formulation, it is second-order accurate, even for non-uniform grids. In the present paper it is shown that when the KBS difference system is reduced to an equivalent three-point system, the governing equations are in fact

*The two-point KBS development reproduced herein is equivalent to the original half-point difference version of the KBS only for equations in divergence form. This point is not clearly expressed by Wornom [5] in his applications of the KBS, see Section 7.

only satisfied to first-order in the usual finite-difference sense; i.e., when the truncation error is evaluated at the mesh point, η_j. It is further shown that, for a model equation, a simpler procedure for obtaining the tridiagonal form of the KBS equations can be derived from the MFE integral method, or alternatively a three-point weighted averaging procedure. These lead directly to a single tridiagonal structure, for a second-order differential equation, in place of the 2x2 block-tridiagonal system resulting from the usual KBS formulation. The MFE approach is extended to higher order by using cubic splines and it is shown that this leads to the fourth-order version of the KBS procedure recently applied by Wornom[5].

Boundary layer solutions using conventional second-order finite-difference collocation, the second and fourth order KBS, and fourth-order spline collocation[3,9] or MFE methods are compared. Conservation and non-conservation form are considered. Finally, the extension of the MFE formulation to three-coordinate parabolic systems and to the transonic small disturbance equations is briefly described. Solutions with this development have not yet been obtained.

2. Keller Box Scheme: Conservation Equations

Let us apply the KBS method to the following ordinary differential equation written in divergence form and with a source term F:

$$(au_\eta)_\eta = F + (bu)_\eta \quad ; \; a = a\,(u_\eta, u; \eta); \; F = F(u; \eta) \qquad (1)$$

$$b = b\,(u; \eta).$$

The KBS method considers (1) as two first-order equations, so that,

$$\left.\begin{aligned} u_\eta &= m \\ (am)_\eta &= F + (bu)_\eta \end{aligned}\right\} \qquad (2a)$$

The KBS finite difference analog of (2a) is:

$$\left.\begin{aligned} u_j-u_{j-1} &= \frac{h_j}{2}(m_j+m_{j-1}) \\ (am)_j-(am)_{j-1} &= \frac{h_j}{2}(F_j+F_{j-1}) + (bu)_j-(bu)_{j-1} \end{aligned}\right\} \quad (2b)$$

where $h_j = \eta_j - \eta_{j-1}$.

Note that the right-hand side of (2b) is simply the trapezoidal rule applied to $\int_{\eta_{j-1}}^{\eta_j} F d\eta$. The error in (2b) at $(j-\frac{1}{2})$ is formally $O(h_j^2)$.

This discretization can be interpreted as the application of a linear polynomial for the integrand. Other polynomial forms will lead to different formulations and can provide higher-order extensions of the KBS method. In a subsequent section we will derive one such scheme and show its equivalence to the fourth-order KBS applied by Wornom.

2.1 Reduction of KBS to Tridiagonal Form.

Let us eliminate the intermediate function m_j from (2b). Solving for m_j and m_{j-1}, we find on $[\eta_{j-1}, \eta_j]$,

$$(a_j+a_{j-1})m_j = \frac{2a_{j-1}}{h_j}(u_j-u_{j-1}) + \frac{h_j}{2}(F_j+F_{j-1}) \quad (3a)$$

$$+ (bu)_j-(bu)_{j-1}$$

$$(a_j+a_{j-1})m_{j-1} = \frac{2a_j}{h_j}(u_j-u_{j-1}) - \frac{h_j}{2}(F_j+F_{j-1}) - (bu)_j+(bu_{j-1}) \quad (3b)$$

Similar expressions for m_j and m_{j+1} are found on $[\eta_j, \eta_{j+1}]$ by incrementing j. After some elimination we find

$$\frac{2a_{j+1}}{h_{j+1}}(u_{j+1}-u_j) - \frac{2a_{j-1}}{h_j}(u_j-u_{j-1}) - \frac{h_{j+1}}{2}(F_{j+1}+F_j) - \frac{h_j}{2}(F_j+F_{j-1})$$

$$-(bu)_{j+1}+(bu)_{j-1} = (a_{j+1}-a_{j-1})m_j \quad (4a)$$

With $\frac{h_{j+1}}{h_j} = \sigma$, b = constant and a_j = constant = 1, we see that

$$\frac{2}{\sigma(1+\sigma)h_j^2}[u_{j+1}-(1+\sigma)u_j+\sigma u_{j-1}] = \frac{\sigma F_{j+1}+(1+\sigma)F_j+F_{j-1}}{2(1+\sigma)} \tag{4b}$$

$$+b(u_{j+1}-u_{j-1})/(1+\sigma)h_j$$

It should be noted that for non-uniform grids ($\sigma \neq 1$), both $u_{\eta\eta}$ and u_η in (1) are represented by first-order accurate finite-difference discretizations. If the truncation error is formally evaluated, at η_j, this three-point analog of the KBS method is in fact only "first-order accurate". The most interesting aspect of equation (4b) is the weighted average of the source term F; this implies a certain amount of smoothing not found in a differential collocation method. This three-point reduction has previously been used by Ackerberg [(6)] and others. It has not been extensively applied and appears to be more efficient than the 2x2 KBS solution procedure. For variable a_j, as in the case of turbulent flows, m_j appears explicitly in (4a) and can be eliminated with (3a). The tridiagonal system is more complex.

3. Modified Finite-Element Formulation (MFE)

Equation (4b) can be derived directly by integrating equation (1) over two adjacent intervals so that,

$$\int_{\eta_{j-1}}^{\eta_j} [u_{\eta\eta}-(bu)_\eta-F]d\eta + \int_{\eta_j}^{\eta_{j+1}} [u_{\eta\eta}-(bu)_\eta-F]d\eta=0 \quad ; \; m = u_\eta \tag{5}$$

The variables $u(\eta)$ and $F(\eta)$ are then approximated by a polynomial over each interval $[\eta_{j-1}, \eta_j]$, j=1, ...,N. For example, with a quadratic polynomial approximation for u over each of the intervals, and a linear polynomial consistent with the trapezoidal rule of integration, for the source term F*, we obtain on $[\eta_{j-1}, \eta_j]$:

*A lower order polynomial for F is consistent with a quadratic approximation for derivatives. The quadratic polynomial in (6) has been described earlier in reference (2) and is designated as $S(\eta;2,1)$.

$$
\left.
\begin{aligned}
u(\eta) &= u_j t + u_{j-1}(1-t) + (u_j - u_{j-1} - h_j m_j)t(1-t) \\
F(\eta) &= F_j t + F_{j-1}(1-t) \\
\text{where } t &= \frac{\eta - \eta_{j-1}}{h_j}
\end{aligned}
\right] \quad (6)
$$

All functional values appearing in (5) are assumed to be individually continuous, i.e., u,F,m.

Completing the integration in equation (5), we obtain

$$(m_{j+1}-m_{j-1})-b(u_{j+1}-u_{j-1})+\frac{h_j}{2}\left[\sigma F_{j+1}+(1+\sigma)F_j+F_{j-1}\right]=0 \qquad (7a)$$

Differentiating (6) we recover the expression in (2b)

$$u_j - u_{j-1} = \frac{h}{2}(m_j+m_{j-1}) \qquad (7b)$$

The expression (7b) is also obtained by imposing continuity of m_j. Eliminating m_j from (7a,7b) we find

$$\frac{2}{\sigma(1+\sigma)h_j^{\,2}}\left[u_{j+1}-(1+\sigma)u_{j-1}\right]=b\,\frac{u_{j+1}-u_{j-1}}{(1+\sigma)h_j} \qquad (7c)$$

$$-(\sigma F_{j+1}+(1+\sigma)F_j+F_{j-1})/(2(1+\sigma))$$

This equation is exactly the same as (4b) and as mentioned previously has the advantage of being of a single tridiagonal form. This system can more easily be solved by the usual two pass algorithm than can the 2x2 block inversion for the KBS method and provides identical solutions. For $F \equiv 0$, (7c) is equivalent to the difference form of the momentum equation in the Davis Coupled Scheme [7].

This modified finite-element formulation differs from the usual finite-element method in that the weighting functions are unity and <u>polynomials are prescribed for each term in the equation rather than for each variable</u>. This leads to a considerable simplification and provides consistent error estimates for the method. Finally, this procedure also applies for a variable coefficients a_j, b_j; in which case (4a) is recovered; it can be made more accurate by considering higher-order polynomials. This is shown in Section 6. The polynomial for $u(\eta)$ in (6) leads to the usual finite-difference relationships if instead of (7a), we require the continuity of $u_{\eta\eta}$. This leads to $S(\eta;2,0)$ of reference (2). The MFE approach does not require this additional continuity condition.

4. Weighted Averaging

The usual three-point differencing for first and second derivatives is given by:

$$u_\eta = \frac{1}{\sigma(1+\sigma)h_j}\left[u_{j+1}+(\sigma^2-1)u_j-\sigma^2 u_{j-1}\right) \tag{8a}$$

and

$$u_{\eta\eta} = \frac{2}{\sigma(1+\sigma)h_j^2}\left[u_{j+1}-(1+\sigma)u_j+\sigma u_{j-1}\right] \tag{8b}$$

In terms of the mesh half-points, these difference formulas can be interpreted by the following weighted averages:

$$u_\eta = m_j = \frac{m_{j+\frac{1}{2}}+\sigma m_{j-\frac{1}{2}}}{(1+\sigma)} \tag{9a}$$

$$u_{\eta\eta} = M_j = \frac{M_{j+\frac{1}{2}}+\sigma M_{j-\frac{1}{2}}}{1+\sigma} \tag{9b}$$

where

$$\left.\begin{aligned} m_{j+\frac{1}{2}} &= \frac{u_{j+1}-u_j}{h_{j+1}} \\ m_{j-\frac{1}{2}} &= \frac{u_j-u_{j-1}}{h_j} \end{aligned}\right] \tag{10}$$

and

$$M_{j+\frac{1}{2}} = \frac{m_{j+1}-m_j}{h_{j+1}} \quad ; \quad M_{j-\frac{1}{2}} = \frac{m_j-m_{j-1}}{h_j} \tag{11}$$

If weighted average consistent with the trapezoidal rule are assumed in lieu of (9), we have the following formulas for m_j, M_j:

$$m_j = u_\eta = \frac{\sigma m_{j+\frac{1}{2}}+m_{j-\frac{1}{2}}}{(1+\sigma)} \tag{12}$$

$$M_j = u_{\eta\eta} = \frac{\sigma M_{j+\frac{1}{2}}+M_{j-\frac{1}{2}}}{1+\sigma}$$

With the same averaging procedure for all source terms, and the additional conditions (10) and (11), we recover the tridiagonal form of the KBS or modified finite-element method, equation (4). It is interesting that as with the modified finite element approach, the weighted averaging here is applied to the governing equation and not the individual variable.

If the expression (9) for M_j is left in terms of m_j (from (11)), the 2x2 $(u,m)_j$ system is truly second-order accurate for both m_j and M_j; the trapezoidal averaging is only first-order accurate. Solutions using this formulation have never been considered.

As a final note, we have conducted some sample boundary layer calculations, both laminar and turbulent, with and without a pressure gradient. These results are shown in Section 7. It is significant that neither the downstream weighted formulation (KBS or MFE), nor the upstream weighted formulation (usual three-point differencing) is superior throughout. The preferred method depends on both the flow profile, and significantly, the finite-difference grid. Therefore, once again no single formulation can be interpreted as a panacea, even for a limited class of flows, i.e., simple boundary layers.

5. Fourth-Order Keller Box Scheme

Recently Wornom [5] has used the fourth-order extension of the KBS proposed by Keller [4]. The extension of the usual trapezoidal rule to fourth-order gives the following two-point Taylor series expansion:

$$g_j - g_{j-1} = \frac{h_j}{2}(g_j' + g_{j-1}') - \frac{h_j^2}{12}(g_j'' - g_{j-1}'') + 0(h_j^5) \tag{13}$$

When applied to equations (2a), we find

$$\left.\begin{aligned} u_j - u_{j-1} &= \frac{h_j}{2}(m_j + m_{j-1}) - \frac{h_j^2}{12}(M_j - M_{j-1}) && (14a) \\ (am)_j - (am)_{j-1} &= (bu)_j - (bu)_{j-1} + \frac{h_j}{2}(F_j + F_{j-1}) && (14b) \\ &\quad - \frac{h_j^2}{12}(F_j' - F_{j-1}') \end{aligned}\right]$$

The primes denote differentiation with respect to η. The expression (14a) is equivalent to one of the spline collocation formulas, see references (1-3) and Section 6. As applied by Wornom [5], the system (14) is closed by evaluating M_j directly from the governing equations. This results in a 2x2 block for $(u,m)_j$. An alternate procedure shown in the next section will be to apply the spline relationships between m_j and M_j.

If we increment j to j+1 and add the resulting equations to (14), we obtain a three point version of this scheme. The resulting equations are as follows:

$$a_{j+1}m_{j+1} - a_{j-1}m_{j-1} = (bu)_{j+1} - (bu)_{j-1} + \frac{h_j}{2}[\sigma F_{j+1} + (1+\sigma)F_j + F_{j-1}]$$
$$-\frac{h_j^2}{12}[\sigma^2(F_{j+1}' - F_j') + (F_j' - F_{j-1}')] \tag{15}$$

$$u_{j+1}-u_{j-1} = \frac{h_j}{2}[\sigma m_{j+1}+(1+\sigma)m_j+m_{j-1}] \tag{16}$$

$$\frac{-h_j^2}{12}[\sigma^2(M_{j+1}-M_j)+(M_j-M_{j-1})]$$

5.1 Keller Box Schemes: Non-Divergence Equations

If we neglect the higher order terms in the two-point formula (13), we have

$$g_j-g_{j-1} = \frac{h_j}{2}(g_j'+g_{j-1}') \tag{17}$$

As noted earlier, this is equivalent to the application of the trapezoidal rule and when applied to equation (2a) leads to the second-order KBS. However, the application of the above algorithm to the non-divergence form of the equations does not lead to the original KBS method [8]. Equations (2a), with a = 1, can be re-written in non-divergence form as

$$u_\eta = m \tag{18a}$$

$$m_\eta = bm+\bar{F} \quad \text{where } \bar{F} = F+b_\eta u. \tag{18b}$$

Application of (17) to this set of equations leads to

$$u_j-u_{j-1} = \frac{h_j}{2}(m_j+m_{j-1}) \tag{19a}$$

and

$$m_j-m_{j-1} = \frac{h_j}{2}(b_j m_j+b_{j-1}m_{j-1}+\bar{F}_j+\bar{F}_{j-1}) \tag{19b}$$

Using half-point differencing, as is done in the original KBS method, the equation (18b) becomes

$$m_j-m_{j-1} = \frac{h_j}{2}\left[\left(\frac{b_j+b_{j-1}}{2}\right)(m_j+m_{j-1})+\bar{F}_j+\bar{F}_{j-1}\right] \tag{19c}$$

It is easily seen that equations (19b) and (19c) are identical only for constant b, i.e., divergence form. The calculations of section 7 will show that in certain cases there can be a significant increase in accuracy when applying (19c) in place of (19b). The reduction to tridiagonal form follows the same course and for variable grids the system remains first-order accurate. For divergence form equations the half-point differences and two-point series formulas are identical. The accuracy of the fourth-order two-point formula (13) for non-divergence equations has not been established as Wornom [5] only considers the divergence form equations characterized by (14b). If non-divergence form is considered, i.e., (18b), fourth-order accurate half-point differencing gives

$$m_j-m_{j-1} = \frac{h_j}{2}\left[\left(\frac{b_j+b_{j-1}}{2}\right)(m_j+m_{j-1})+\bar{F}_j+\bar{F}_{j-1}\right] - \frac{h_j^2}{12}\left[F_j'-F_{j-1}'\right]$$
$$-\frac{h_j^2}{48}\left[3(b_j-b_{j-1})'(m_j+m_{j-1})+3(m_j-m_{j-1})'(b_j+b_{j-1})\right.$$
$$\left.-2(bm)_j'+2(bm)_{j-1}'\right] \tag{20a}$$

and (13) leads to

$$m_j-m_{j-1} = \frac{h_j}{2}\left[b_jm_j-b_{j-1}m_{j-1}+\bar{F}_j+\bar{F}_{j-1}\right] \tag{20b}$$
$$-\frac{h_j^2}{12}\left[(bm+\bar{F})_j' - (bm+\bar{F})_{j-1}'\right]$$

Once again (20a) equals (20b) only for constant b.

6. Higher-Order Extension of the Modified Finite-Element Method.

The formulation of section 3 provides a consistent approach for deriving more accurate integral approximations. This can be achieved by using higher-order polynomials for F and u in (6). We consider here a cubic spline polynomial. Spline approximations have been used earlier [1-3] in differential collocation procedures. The cubic polynomial approximation over the interval (η_{j-1}, η_j) can be written as:[1-3]

$$S(\eta;3,1) = S(\eta) = h_j m_{j-1} t(1-t) - h_j m_j t^2 (1-t) + u_{j-1}(1+2t)(1-t)^2 + u_j t^2 (3-2t) \tag{21}$$

From (21) a variety of spline relationships can be derived [1-3]; among these are

$$M_j = \frac{2}{h_j}(m_{j-1}+2m_j) - 6\,\frac{u_j-u_{j-1}}{h_j^{\,2}} \tag{22a}$$

$$M_j = -\frac{2}{h_j}(2m_j+m_{j+1}) + 6\,\frac{u_{j+1}-u_j}{h_j^{\,2}} \tag{22b}$$

Decrementing j to j-1 in (22) and subtracting the resulting equation from (22a), we recover the two-point expression (13),

$$u_j-u_{j-1} = \frac{h_j}{2}(m_j+m_{j-1}) - \frac{h_j^{\,2}}{12}(M_j-M_{j-1}) \quad , \tag{23}$$

which forms the basis of the fourth-order extension of the KBS.

Direct integration of equation (1) over two adjacent intervals, with the polynomial expression (21) for source terms, leads to

$$\begin{aligned} a_{j+1}m_{j+1}-a_{j-1}m_{j-1} - (bu)_{j+1} + (bu)_{j-1} \\ = \frac{h_j}{2}[\sigma F_{j+1}+(1+\sigma)F_j+F_{j-1}] \\ - \frac{h_j^{\,2}}{12}[\sigma^2(F'_{j+1}-F'_j)+(F'_j-F'_{j-1})] \end{aligned} \tag{24}$$

From the spline equations (22a) and (22b) the following three-point formula is obtained:

$$\begin{aligned} u_{j+1}-u_{j-1} = \frac{h_j}{2}[\sigma m_{j+1}+(1+\sigma)m_j+m_{j-1}] \\ - \frac{h_j^{\,2}}{12}[\sigma^2(M_{j+1}-M_j)+(M_j-M_{j-1})] \end{aligned} \tag{25}$$

Equations (24) and (25) are identical with the fourth-order extension of the KBS applied by Wornom (see equations (15) and (16)). Wornom eliminates M_j from (14) with the non-divergence form of the equation (18b); in the present formulation this is unnecessary as M_j is determined from the spline relationships (22). A small increase in the truncation error is introduced with (22), but the substitution of (18b), required by Wornom, is avoided. This can be extremely important for more complex equations and/or multi-dimensional flows.

As a final consideration the simple quintic spline MFE is developed. This has earlier [3] been designated as $S(\eta;5,1)$ and leads to a sixth order formulation:

$$
\begin{aligned}
S(\eta;5,1) = S(\eta) = {} & u_{j-1} + h_j m_{j-1}\, t + \frac{h_j^2}{2} M_{j-1}\, t^2 \\
& + [10(u_j-u_{j-1}) - h_j(4m_j+6m_{j-1}) + \frac{h_j^2}{2}(M_j-3M_{j-1})]\, t^3 \\
& - [15(u_j-u_{j-1}) - h_j(7m_j+8m_{j-1}) + \frac{h_j^2}{2}(2M_j-3M_{j-1})]\, t^4 \\
& + [6(u_j-u_{j-1}) - 3h_j(m_j+m_{j-1}) + \frac{h_j^2}{2}(M_j-M_{j-1})]\, t^5
\end{aligned}
\tag{26}
$$

With the polynomial (26) we integrate (2a) over the interval $[\eta_{j-1}, \eta_j]$ to obtain the sixth-order extension of the two-point Keller formula (13). We find

$$
\begin{aligned}
g_j - g_{j-1} = {} & \frac{h_j}{2}(g_j' + g_{j-1}') - \frac{h_j^2}{10}(g_j'' - g_{j-1}'') \\
& + \frac{h_j^3}{120}(g_j''' + g_{j-1}''') + O(h_j^7)
\end{aligned}
\tag{27}
$$

If we complete the integration (5) using (26) for F, the sixth-order MFE formula for (2a) becomes

$$a_{j+1}m_{j+1}-a_{j-1}m_{j-1}=(bu)_{j+1}-(bu)_{j-1}+\frac{h_j}{2}\left[\sigma F_{j+1}+(1+\sigma)F_j+F_{j-1}\right]$$

$$-\frac{h_j^2}{10}\left[\sigma^2F'_{j+1}-(\sigma^2-1)F'_j-F'_{j-1}\right] \tag{28}$$

$$+\frac{h_j^3}{120}\left[\sigma^3F''_{j+1}+(\sigma^3+1)F''_j+F''_{j-1}\right]$$

In addition the spline formulas from reference 3 are

$$7m_{j+1}+8(1+\sigma^3)m_j+7\sigma^3m_{j-1}=\frac{15}{\sigma h_j}\left(u_{j+1}+(\sigma^4-1)u_j-\sigma^4u_{j-1}\right) \tag{29a}$$

and

$$-M_{j+1}+3(1+\sigma)M_j-\sigma M_{j-1}=\frac{20}{\sigma^2h_j^2}\left(u_{j+1}-(1+\sigma^3)u_j+\sigma^3u_{j-1}\right)$$

$$+\frac{4}{h_j\sigma}\left(-2m_{j+1}+3(\sigma^2-1)m_j+2\sigma^2m_{j-1}\right) \tag{29b}$$

Since $F = F(u;\eta)$, the F' and F'' terms in (28a) are at most functions of u,m,M. The equations (28), (29a) and (29b) provide a (3x3) block-tridiagonal system for $(u,m,M)_j$. We recall that the second-order MFE is a scalar system for u_j, and the fourth-order MFE is 2x2 for $(u,m)_j$. The M_j terms in (28,29) can be eliminated by using the differential spline approximation to (1) in non-divergence form. This is the same procedure used by Wornom to eliminate M_j in the fourth-order KBS development. This was unnecessary in the fourth-order MFE formulation, as the spline relationships (22) were available to evaluate M_j and maintain a (2x2) system. If a (2x2) sixth-order MFE system is preferred, it is necessary to apply the non-divergence M_j elimination; however, as discussed earlier this is not desirable for more complex systems of equations.

7. Two dimensional MFE

In this section, the modified finite element method is applied with a two-dimensional equation of the following type:

$$T_t + (uT)_x + (vT)_y = \frac{1}{R_e} (T_{xx}+T_{yy}) + q \qquad (30)$$

If we define

$$\left.\begin{aligned} F &= uT - \frac{1}{R_e} T_x \\ G &= VT - \frac{1}{R_e} T_y \\ P &= q - T_t \quad , \end{aligned}\right] \qquad (31)$$

The above equation can be written as:

$$F_x + G_y = P \qquad (32)$$

In this two-dimensional case the integration is over a rectangular grid. The method can be used for other element shapes; however, we shall only consider a rectangular element in the present investigation. Bicubic spline polynomials can be used for curve fitting in two dimensions; however, in the present analyses, partial splines [11] are used for discretizing equation (32). This amounts to a splitting procedure, where the curve fit is made in the x direction and then a second curve fit is made in the y direction; the order is arbitrary. This procedure has been shown to be equivalent to the use of bicubic splines (see reference 11).

Integrating equation (32) between (x_{i-1}, x_{i+1}), we obtain

$$\begin{aligned} F_{i+1}-F_{i-1} &+ \Big[\frac{k_i}{2} \left\{\sigma_x G_{i+1} + (1+\sigma_x)G_i+G_{i-1}\right\} \\ &- \frac{k_i^{\,2}}{12} \left\{\sigma_x^{\,2} \left\{(G_x)_{i+1}-(G_x)_i\right\} + ((G_x)_i-(G_x)_{i-1})\right\}\Big] \qquad (33a) \\ &= S_i-Z_i \end{aligned}$$

where $\quad k_i = x_i - x_{i-1} \quad ; \quad \sigma_x = k_{i+1}/k_i$

$$S_i = \frac{k_i}{2}\,(\sigma_x P_{i+1} + (1+\sigma_x) P_i + P_{i-1}) \tag{33b}$$

$$Z_i = \frac{k_i^{\,2}}{12}\,[\sigma_x^{\,2}\left\{(P_x)_{i+1} - (P_x)_i\right\} + \left\{(P_x)_i - (P_x)_{i-1}\right\}]$$

Integrating over $[y_{j-1}, y_{j+1}]$, with $h_j = y_j - y_{j-1}$ and $\sigma_y = h_{j+1}/h_j$, we obtain

$$\begin{aligned}
&\frac{h_j}{2}\,[\sigma_y(F_{i+1,j+1} - F_{i-1,j+1}) + (1+\sigma_y)(F_{i+1,j} - F_{i-1,j}) \\
&\quad + (F_{i+1,j-1} - F_{i-1,j-1})] \\
&\frac{-h_j^{\,2}}{12}\,[\sigma_y^{\,2}\left\{(F_y)_{i+1,j+1} - (F_y)_{i-1,j+1} - (F_y)_{i+1,j} + (F_y)_{i-1,j}\right\} \\
&\quad + \left\{(F_y)_{i+1,j} - (F_y)_{i-1,j} - (F_y)_{i+1,j-1} + (F_y)_{i-1,j-1}\right\}] \\
&+ \frac{k_i}{2}\,[\sigma_x(G_{i+1,j+1} - G_{i+1,j-1}) + (1+\sigma_x)(G_{i,j+1} - G_{i,j-1}) \\
&\quad + (G_{i-1,j+1} - G_{i-1,j-1})] \\
&- \frac{k_i^{\,2}}{12}\,[\sigma_x^{\,2}\left\{(G_x)_{i+1,j+1} - (G_x)_{i,j+1} - (G_x)_{i+1,j-1} + (G_x)_{i,j-1}\right\} \\
&\quad + \left\{(G_x)_{i,j+1} - (G_x)_{i-1,j+1} - (G_x)_{i,j-1} + (G_x)_{i-1,j-1}\right\}] \\
&= \frac{h_j}{2}\,[\sigma_y(S_{i,j+1} - Z_{i,j+1}) + (1+\sigma_y)\,(S_{i,j} - Z_{i,j}) + (S_{i,j-1} - Z_{i,j-1})] \\
&\quad -h_j^2/12\,[\sigma_y^{\,2}\left\{(S_y)_{i,j+1} - (Z_y)_{i,j+1} - (S_y)_{i,j} + (Z_y)_{i,j}\right\} \\
&\quad + \left\{(S_y)_{i,j} - (Z_y)_{i,j} - (S_y)_{i,j-1} + (Z_y)_{i,j-1}\right\}]
\end{aligned} \tag{34}$$

7.1 Example: Second-order MFE for Transonic Small Disturbance Equations

As an example of the two dimensional MFE formulation, the transonic small disturbance equations are considered. In this analysis only second-order accurate theory is discussed; i.e., the k_i^2 and h_j^2 terms in the previous development are neglected. The fourth order extension is not considered here although it can be obtained from the equations of section 7. We include this example here since the earlier formulation [13] of this problem is only first-order accurate in the supersonic region.

The transonic small disturbance equations are:

$$[k - (\gamma+1)\Phi_x]\Phi_{xx} + \Phi_{yy} = 0 ,$$

where γ is the ratio of specific heats, k the transonic similarity parameter and Φ the velocity potential:

$$u = \Phi_x \text{ and } v = \Phi_y .$$

Rewriting the equation in conservation form, we obtain

$$[ku - \frac{\gamma+1}{2} u^2]_x + v_y = 0 .$$

This equation is identical to equation (32) with

$$F = ku - \frac{\gamma+1}{2} u^2 ,$$

and

$$G = v ,$$

$$P = 0 .$$

Since the equation is of mixed type, the integration must reflect the characteristic domain of dependence in the supersonic region. Therefore for subsonic regions we integrate over $[x_{i-1},x_{i+1}],[y_{j-1},y_{j+1}]$ so that (34) applies directly. For the supersonic domains we integrate over $[x_{i-2}, x_i]$, $[y_{j-1}, y_{j+1}]$ so that (34) applies with $i \rightarrow i-1$. For the full potential equations a similar procedure would apply.

In order to show the second-order accuracy for the supersonic regions, equation (33a) with $i \rightarrow i-1$ becomes, in terms of (u,v),

$$[ku - \frac{\gamma+1}{2}u^2]_i - [ku - \frac{\gamma+1}{2}u^2]_{i-2} = \frac{(\Delta x)_{i-1}}{2} [\sigma_x (v_y)_i + (1+\sigma_x)(v_y)_{i-1} + (v_y)_{i-2}]$$

where $\sigma_x = (\Delta x)_i / (\Delta x)_{i-1}$.

It is easily shown that this equation is second-order accurate in Δx even for non-uniform grids. The y integration then leads to (34). The final expression for (u,v) or Φ involves the nine points on $[x_{i-2}, x_i]$, $[y_{j-1}, y_{j+1}]$, where the Murman-Cole [13] first-order formula requires only five points.

8. Boundary Layer Examples

The fourth-order MFE method is applied to the parabolic boundary layer equations written in conservation form. Comparisons are made with second-order techniques, the fourth-order spline 4 collocation method applied to the non-divergence differential form of the equations and the Wornom fourth-order adaption of the KBS. Specific examples include (1) the similarity (ordinary differential) equations governing the laminar flow over a flat plate and at a stagnation point, (2) the quasi-similar model equation for the turbulent flat plate boundary layer, and (3) the non-similar boundary layer, both laminar and turbulent, described by a decelerating linear external velocity field (Howarth problem). In almost all cases an exact or very accurate numerical solution is available for comparison purposes.

The governing boundary layer equations are

$$\left((1+\varepsilon)V_\eta\right)_\eta + \left((f+2\xi f_\xi)V\right)_\eta = V^2-\beta(1-V^2)+2\xi V_\xi^2 \quad , \tag{35a}$$

$$f_\eta = f_\eta(\xi,\eta) = V\ (\xi,\eta) = V \quad , \tag{35b}$$

where ε is the turbulent eddy viscosity defined here by the Michel [9] single layer model

$$\varepsilon = R_e^{3/2}F^2\ell^2|u_\eta|/(2\xi)^{\frac{1}{2}} \quad ;$$

$$F = 1-\exp[-\eta/A]; \ A = 26\ [(2\xi R_e)^{\frac{1}{2}}(u_\eta)_{\eta=0}]^{-\frac{1}{2}} \quad ;$$

$$\ell = 0.085\ \eta_e(2\xi/R_e)^{\frac{1}{2}}[\tanh(0.41\eta/0.085\eta_e)] \quad ;$$

and $\eta = y(R_e/2\xi)^{\frac{1}{2}}$; $\xi = \int_0^x u_e dx$

$\beta(\xi) = \dfrac{2\xi \frac{du_e}{dx}}{u_e^2}$; $\eta_e = \eta$ at $V = 0.995$ and e denotes the boundary layer edge.

For the flat plate geometry, $\beta=0$; for the stagnation point, $\beta=1$; for the Howarth flow, $u_e = 1-x$ and $\beta = -2\xi/(1-2\xi)^{\frac{1}{2}}$; for laminar flow, $\varepsilon = 0$.

The boundary conditions are

$$f(\xi,0) = V(\xi,0) = 0 \text{ and } \lim_{\eta\to\infty} V(\xi,\eta) = 1 \quad . \tag{35c}$$

8.1 Fourth-Order MFE Equations

For the boundary layer flow (35) the MFE equations are given by (22-25) with $a = 1+\xi$, $b = -(f+2\xi f_\xi)$ and $F = V^2-\beta(1-V^2) + 2\xi V_\xi^2$ for (35a), and $a = 0$, $b = 1$, $F = V$ for (35b). The streamwise gradients are discretized by backward or central differences. A 3x3 block-tridiagonal system for (u,M,f) results.

The boundary conditions are given by (35c); in addition, the boundary values for m or M are given by (14b), or (14b) and (22), respectively. An alternate boundary condition, used earlier [10] for differential spline boundary layer calculations is given by

$$M(\xi,0)^* = -\beta - h_2{}^2\big((2\beta-1)m^2(\xi,0)+\xi m_\xi{}^2(\xi,0)\big)/12 = -\beta\, h_2^2 V^{iv}(\xi,0)/12.$$

where $h_2 = \eta_2$; η_j denotes the grid point normal to the surface; $h_j = \eta_j - \eta_{j-1}$ is the local mesh width; $\Delta\xi = \xi_i - \xi_{i-1}$ is the mesh width along the surface.

8.2 Solutions

(a) Similar Laminar Boundary Layer

The results for the similarity solutions, $(\frac{\partial}{\partial\xi})=0$ in (35a), for flat plate (β=0) and stagnation point (β=1) flows are given in Tables 1-3 for a variable and uniform mesh, with ten intervals be-

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*$m(0,\xi)$ can also be obtained from a Taylor series expansion at η=0. For coarse grids, this can be preferable to the spline relationship for $m(0,\xi)$[3].

tween the surface and the outer boundary. For the variable grid

$$\eta = a_o \zeta/(1+b_o \zeta)^{\alpha} \tag{36}$$

where $\alpha = 8.26$, $b_o = -0.4$ and $a_o = 24.2538\ (1+b_o)^{\alpha}$.

Figure 1 depicts the percentage error in wall shear, for $\beta = 1$, as a function of the number of intervals. The velocity profiles for the variable grid with $\beta = 0$ are given in Table 1. The KBS4 and fourth-order MFE methods using divergence (conservation) form equations are generally the most accurate. The KBS4 method, which requires the use of the governing equation for M_j is slightly more accurate than MFE; however, the latter is less time consuming and therefore, for equal accuracy the two procedures require approximately the same CPU time. For more complex systems of equations the MFE approach should become more favorable.

The second-order methods are inconclusive. For the flat plate geometry and a variable grid, the KBS is excellent and the finite-difference poor. For $\beta = 0$, with a uniform grid, and for the variable grid $\beta = 1$ solution, the opposite holds; in fact, the non-divergence finite-difference solutions are better than the divergence KBS results. The same is true for the so-called Davis Coupled Scheme [7] (DCS).

The solutions deteriorate, in many cases, when non-conservation equations are considered. The KBS2 results are particularly noteworthy in this respect, moreover, the large differences in the half-point KBS and two-point KBS2 adaption used by Wornom [5] are evident. Non-conservation KBS4 solutions were not available.

The non-divergence spline 4 solutions are poor for the variable grid $\beta = 0$ case but are significantly better for the $\beta = 0$ uniform grid and $\beta = 1$ variable grid examples. Furthermore, when the spline 4 method is applied to the equations transformed by (36), instead of a grid generated by (36), a reasonable improvement results.

It should be noted that the integral approach (KBS,MFE) would involve considerably more arithmetic if (1) the transformed equations were considered, (2) the governing equations were more complex or extensive (in particular, the use of the governing equation for M_j in KBS4 requires more operations), or (3) the equations are two-dimensional; as shown in Section 7, the complexity of the integrated equations is increased significantly. On the other hand, the non-divergence spline 4 procedure is not modified greatly in these cases; therefore, the slight decrease in accuracy may be offset by its simplicity. Finally, the spline 4 procedure can be adapted in order to obtain conservation solutions in differential form; in references (2,3) significant improvements in accuracy were noted. This requires several additional curve fits, but does not appear to be as complicated as the two-dimensional KBS or MFE procedures for simple rectangular elements. For other element shapes, the integral finite-element approach might be more preferable.

In summary, it can be concluded that the accuracy of the results, of any of the fourth-order methods described here, is very dependent on the combination of pressure gradient parameter β and the choice of grid.

(b) Quasi-Similar Turbulent Boundary Layer

For these calculations we assume that $\frac{\partial}{\partial \xi} = 0$ in (35a), even though the solutions do depend on $(R_e \xi)$. This is a fair approximation and allows us to consider an ordinary differential equation for the turbulent boundary layer. In subsequent examples, the full non-similar equations will be considered.

The mesh is given by the grid generated from (36), with $\alpha = -109$, $b_o = 0.05$ and $a_o = 60(1+b_o)^{\alpha}$. The percentage error in shear stress is depicted on Figures 2 and 3, for $\beta = 0$ and $\beta = 0.5$, respectively. For these cases T_e in the turbulent eddy viscosity model is given as 24.5 and 15.8,[5] respectively. The results for the KBS4 and MFE are good, although less than fourth-order accurate. This appears to be caused by the error in the numerical evaluation of the eddy viscosity ε. For fixed ε, fourth-order accuracy should be recovered.

The non-divergence form solutions, obtained with the grid transformation are reasonable, but somewhat erratic. The second-order finite-difference solutions are, surprisingly, more accurate than the divergence KBS2 results. The non-divergence KBS2 solutions are rather poor.

(c) Non-Similar Solutions

For the non-similar boundary layer calculations a, b and F in (35) are defined in section 8.1. Three cases have been considered here: (1) turbulent flow with $\beta = 0.5$; (2) laminar Howarth problem, and (3) turbulent Howarth problem. In the first case the quasi-similar solution has previously been discussed. The effects of the non-similar terms will be examined. For case (2) the separation point has been critically examined in many investigations; it has been found that separation occurs at $x = 0.1198$. In this section, separation point calculations for course grids are compared with earlier solutions [12] obtained with the non-divergence spline 4 system of equations, as well as second-order methods. Finally, the suppression of separation for the turbulent boundary layer in a decreasing linear external flow is described. To the authors' knowledge solutions for this flow have not been published. This case is considered here, simply to demonstrate the applicability of the MFE method to turbulent boundary layers in adverse pressure gradients.

The solutions are shown in Tables 4-6. For constant $\beta = 0.5$ the grid defined by (36) with 10 intervals in η is specified; $\eta_e=15.8$ in the eddy viscosity model. This case has previously been treated by Wornom [5] using the KBS4 procedure. The effect of the marching increment $\Delta\xi$ and comparisons with the quasi-similar solution at $R_{e\xi}=1.88\times10^6$ are given in Table 4. For Wornom's grid the ξ_i are given as: 0, 86×10^{-5}, 43×10^{-4}, 86×10^{-4}, 43×10^{-3}, 0.17, 0.26, 0.34, 0.51, 0.68, 0.86, 1.03, 1.28, 1.50, 1.71 and 1.88 for $i=1,\ldots,17$. There is a small effect of the ξ spacing, but more important is the assumption that $\eta_e=15.8$ throughout. This should only be true at $R_e\xi=1.88\times10^6$, and therefore the final solutions have an inherent error associated with the inaccurate η_e value used for $\xi<1.88$. A more accurate solution* would require the precise evaluation of η_e at each ξ location; however, with a crude η grid this is

*In a more exact calculation with 60 intervals it was found that at $R_e=430{,}000$, $\eta_e\approx10$.

not possible, and this leads us to believe that for the turbulent boundary layer, the error estimates obtained with the model quasi-similar problem are somewhat artificial.

For the laminar Howarth problem a very coarse uniform grid, with h=1.0 provides reasonably accurate estimates of the separation point, see Table 5. The MFE results are somewhat less accurate than those obtained with the non-divergence spline 4 [12] method but both are significantly better than the second-order solutions. Some calculations were also made with the variable grid considered for the similarity solutions. As was found with the similarity problems, the non-divergence spline 4 results become less accurate than those obtained with the uniform grid. Surprisingly, the MFE solutions also deteriorate; and, for a reason that is not clear, these calculations require orders of magnitude more iterations for convergence than do the spline 4 calculations or the MFE solutions with the uniform grid. It would appear that the MFE calculations are somewhat more sensitive to the grid generated by (36) when non-similar flows with adverse pressure gradients are considered. The similarity results using MFE and (36) were extremely accurate and rapidly convergent.

For the turbulent Howarth problem, separation does not occur for $\xi<0.31$ or $R_e\xi<580{,}000$; the η_e value in the eddy viscosity model is evaluated at each ξ location and the transformation (36) with 60 intervals is specified in order to accurately estimate η_e. The MFE results, on Table 6, appear to be reasonable, although the accuracy is difficult to assess. The ξ grid is rather crude, $\Delta\xi=0.005$, and comparisons with other procedures were not possible.

In summary, the accuracy of the various methods described herein appears to be dependent on a combination of (1) the problem, i.e. laminar or turbulent, constant β or variable β, (2), the choice of the grid and (3) non-divergence or divergence form of the equations. Generally, the fourth-order methods lead to improvements over second-order techniques but the relative accuracy of the fourth-order procedure is dependent on the factors just mentioned.

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Table 1: Laminar Flat Plate ($\beta=0$): Variable Grid to 24.2538
(10 intervals)

(a) Surface Shear Stress

Conservation Form Solutions

Exact	MFE	KBS4	KBS2	F.D.
0.469600	0.470362	0.469835	0.468759	0.380908

Non-Conservation Form Solutions

SPLINE4	TRANSFORMED PLANE SPLINE4	KBS	KBS2	DCS
0.438678	0.476971	0.494798	0.421500	0.488898

(b) Velocity Profiles

Conservation Form Solutions

Grid	Exact	KBS4	MFE	KBS2
0.0000	.00000	0.00000	0.0000000	0.00000
0.0500	.02347	.02348	.0235044	.02343
0.1421	.06671	.06674	.0667994	.06659
0.3076	.14437	.14444	.1446144	.14412
0.6023	.28164	.28178	.2819718	.28113
1.1266	.51474	.51498	.5157463	.51364
2.0651	.83286	.83293	.8350900	.83263
3.7657	.99552	.99616	.9935382	1.01397
6.9004	1.00000	.99947	1.0010271	.99643
12.8087	1.00000	.99983	.9995595	1.00320
24.2538	1.00000	1.00000	1.0000000	1.00000

Non-Conservation Form Solutions

SPLINE4	TRANSFORMED PLANE SPLINE4	KBS	KBS2	DCS
0.00000000	0.0000	0.0000	0.00000	0.0000
.02192272	.0246	.0247	.02106	.0244
.06231277	.0699	.0703	.05987	.0694
.13487369	.1513	.1521	.12956	.1503
.26319919	.2949	.2963	.25264	.2936
.48176760	.5396	.5382	.46098	.5393
.78493782	.8605	.8544	.75179	.8929
.93994754	1.0187	1.0156	.97564	1.0820
.96388488	.9999	.9960	1.00253	.9969
.95144079	1.0259	1.0036	.99166	1.0687
1.00000000	1.0000	1.0000	1.00000	1.0000

Table 2: Laminar Flat Plate (β=0): Uniform Grid (h=1)

Exact	(NC)S4	(NC) KBS2	(NC)FD
	(0.24%)	(6.15%)	(1.61%)
0.469600	0.470730	0.440743	0.47718
	MFE2 / KBS2	MFE4	KBS4
	(6.9%)	(0.003%)	Unavilable
	0.437207	0.469614	

Table 3: Laminar Stagnation Point (β=1): Variable Grid to 24.2538 (10 intervals)

Exact	(NC)F.D.	(NC)S4	(NC)DCS
	(1.17%)	(0.141%)	(1.53%)
1.23259	1.24826	1.23084	1.25140
F.D.	DCS	MFE	KBS4
(0.226%)	(2.39%)	(0.042%)	(0.016%)
1.23538	(1.26205)	1.23311	(1.2328)

Table 4: Turbulent Non-Similar Solutions (β=0.5, R_{e_ξ}=1.88x10^6, η_e=15.8)

(a) $C_f \times 10^3$

MODEL	MFE($\Delta\xi$=0.005)	MFE(WORNOM GRID)
3.5473	3.9182	3.9344

(b) VELOCITY PROFILE

GRID	MODEL	MFE($\Delta\xi$=0.005)	MFE(WORNOM GRID)
0.0000	0.0000	0.0000	0.0000
0.0500	0.1715	0.1828	0.1836
0.1421	0.3742	0.3935	0.3948
0.3076	0.5162	0.5387	0.5401
0.6023	0.6052	0.6296	0.6312
1.1266	0.6716	0.6980	0.6997
2.0651	0.7342	0.7624	0.7643
3.7657	0.7971	0.8269	0.8288
6.9004	0.8801	0.9110	0.9130
12.8087	0.9636	0.9810	0.9814
24.2538	1.0000	1.0000	1.0000

Table 5: Laminar Howarth Problem: $x_{separation}$

Exact		(NC) Spline4	MFE4	MFE2 KBS2	F.D.
0.1198	Uniform Grid h=1.0	0.1196	0.1225	0.1374	0.1495
	Variable Grid η_{max}=24.2538	0.1096	0.1328	0.1515	------

Table 6: Turbulent Howarth Problem ($Re=1.88 \times 10^6$)

(a) Skin Friction Coefficent

ξ	0.0410	0.0710	0.1010	0.1310	0.1610
$C_f \times 10^3$	6.1360	5.2861	4.7318	4.2847	3.8721

ξ	0.1910	0.2210	0.2510	0.2810	0.3110
$C_f \times 10^3$	3.4849	3.0593	2.5985	2.0266	1.2409

(b) Velocity Profile $R_{e_\xi}=584680$

$\eta_e=23.3468$

η Grid Location	Velocity Profile
0.0	0.0
0.53678271D-02	0.36250401D-02
0.11755137D-01	0.80003324D-02
0.19305691D-01	0.13258937D-01
0.28181089D-01	0.19559811D-01
0.38562823D-01	0.27093096D-01
0.50654555D-01	0.36086184D-01
0.64684642D-01	0.46808622D-01
0.80908942D-01	0.59573963D-01
0.99613913D-01	0.74729457D-01
0.12112005D 00	0.92622170D-01
0.14578572D 00	0.11351693D 00
0.17401132D 00	0.13746167D 00
0.20624400D 00	0.16413698D 00
0.24298282D 00	0.19282160D 00
0.28478443D 00	0.22254502D 00
0.33226939D 00	0.25235271D 00
0.38612918D 00	0.28148961D 00
0.44713391D 00	0.30947067D 00
0.51614079D 00	0.33603731D 00
0.59410363D 00	0.36111264D 00
0.68208317D 00	0.38472005D 00
0.78125856D 00	0.40696584D 00
0.89294008D 00	0.42797281D 00
0.10185830D 01	0.44790580D 00
0.11598033D 01	0.46690318D 00
0.13183941D 01	0.48514236D 00
0.14963451D 01	0.50274604D 00
0.16958626D 01	0.51989529D 00
0.19193929D 01	0.53668604D 00
0.21696468D 01	0.55330950D 00

η Grid Location	Velocity Profile
0.24496278D 01	0.56982869D 00
0.27626621D 01	0.58645984D 00
0.31124324D 01	0.60322403D 00
0.35030147D 01	0.62038413D 00
0.39389191D 01	0.63790398D 00
0.44251342D 01	0.65612327D 00
0.49671768D 01	0.67492265D 00
0.55711455D 01	0.69475977D 00
0.62437805D 01	0.71538652D 00
0.69925294D 01	0.73742855D 00
0.78256183D 01	0.76042258D 00
0.87521320D 01	0.78520705D 00
0.97821000D 01	0.81093406D 00
0.10926593D 02	0.83866090D 00
0.12197826D 02	0.86683199D 00
0.13609277D 02	0.89663058D 00
0.15175811D 02	0.92521038D 00
0.16913818D 02	0.95358411D 00
0.18841370D 02	0.97561814D 00
0.20978384D 02	0.99451582D 00
0.23346809D 02	0.99975893D 00
0.25970828D 02	0.10000232D 01
0.28877075D 02	0.99999764D 00
0.32094886D 02	0.10000002D 01
0.35656556D 02	0.99999998D 00
0.39597641D 02	0.10000000D 01
0.43957270D 02	0.10000000D 01
0.48778502D 02	0.10000000D 01
0.54108710D 02	0.10000000D 01
0.60000000D 02	0.10000000D 01

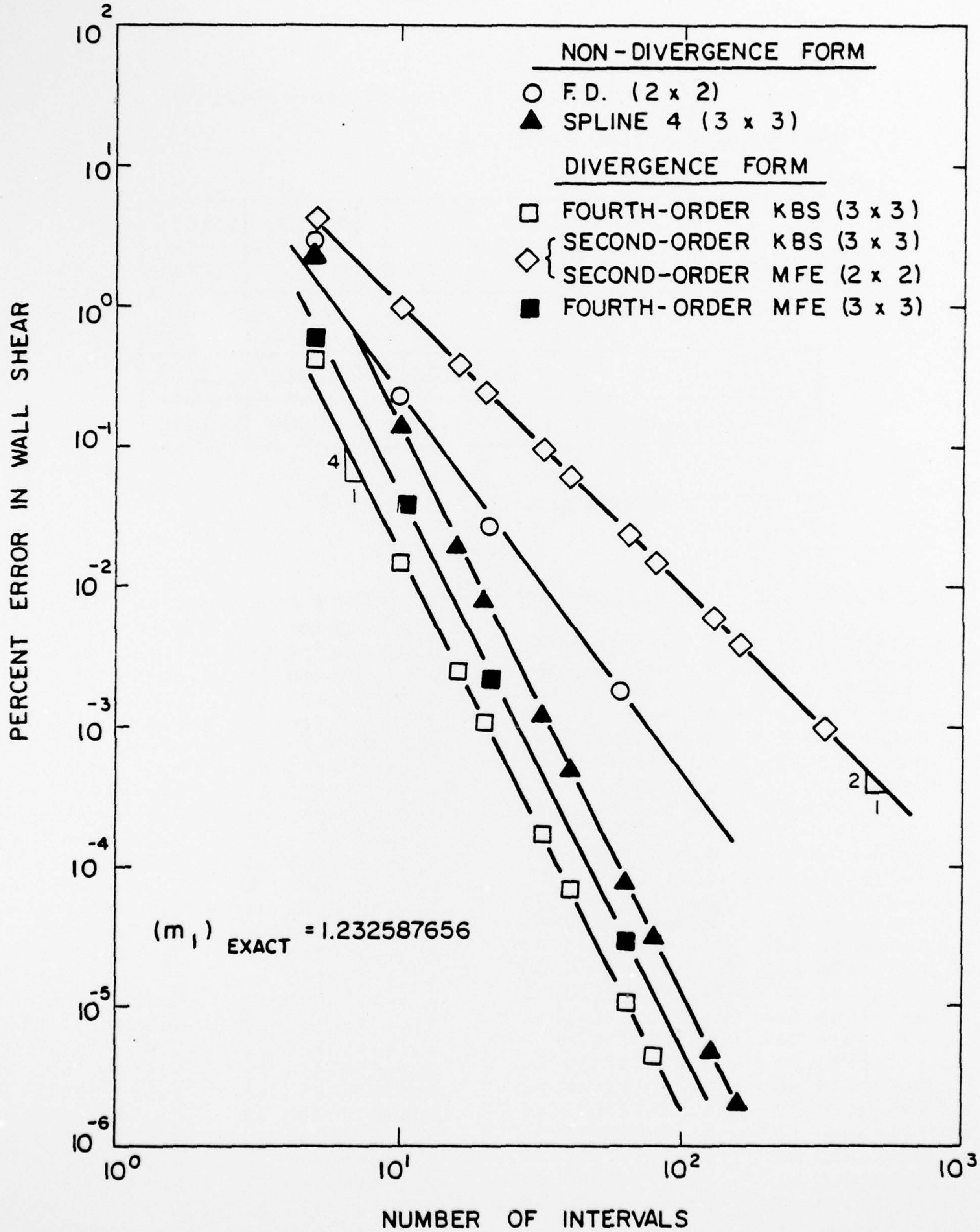

FIG.1 PERCENT ERROR IN THE WALL SHEAR: STAGNATION POINT FLOW ($\beta = 1$)

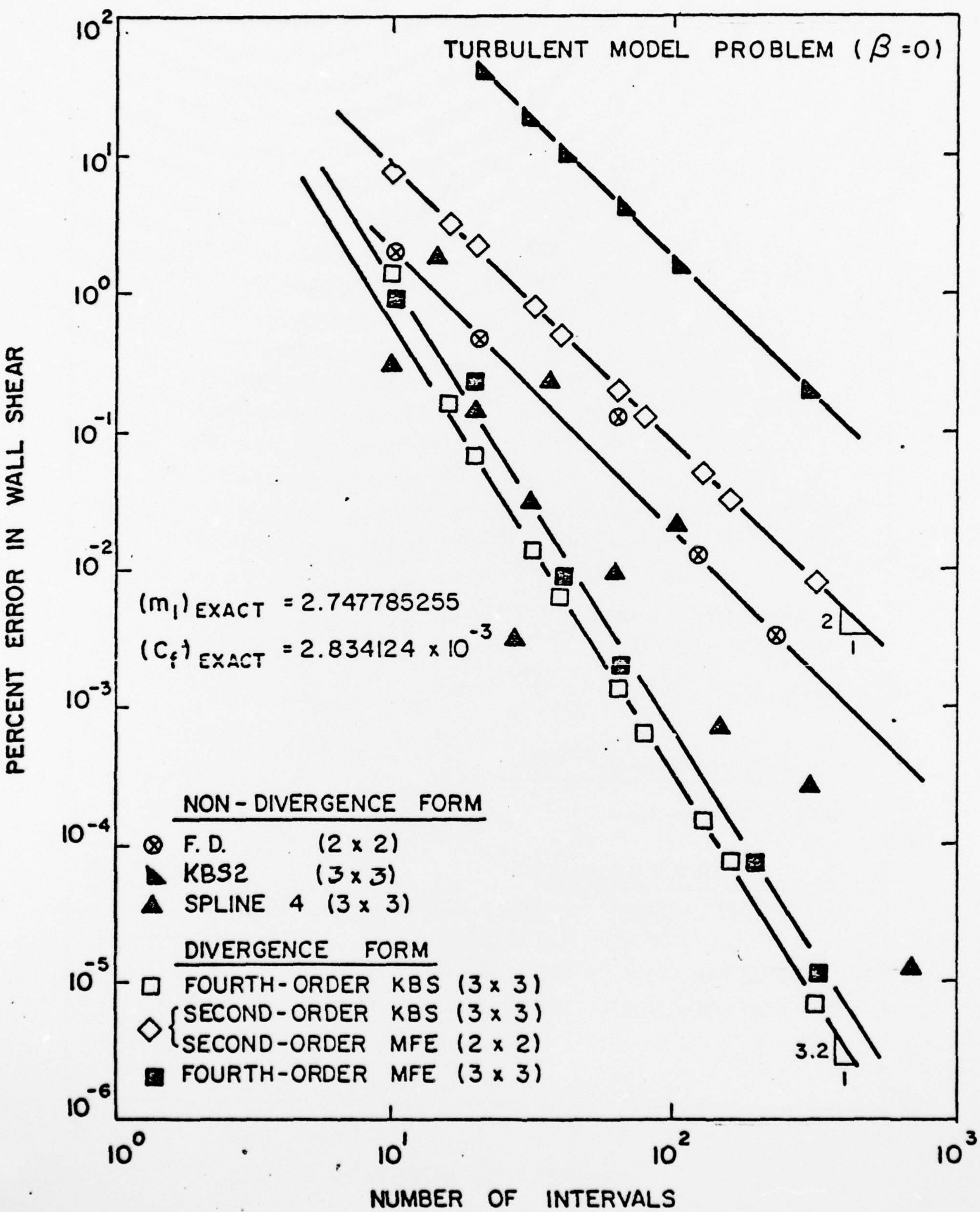

FIG. 2 PERCENT ERROR IN THE WALL SHEAR FOR MODEL PROBLEM

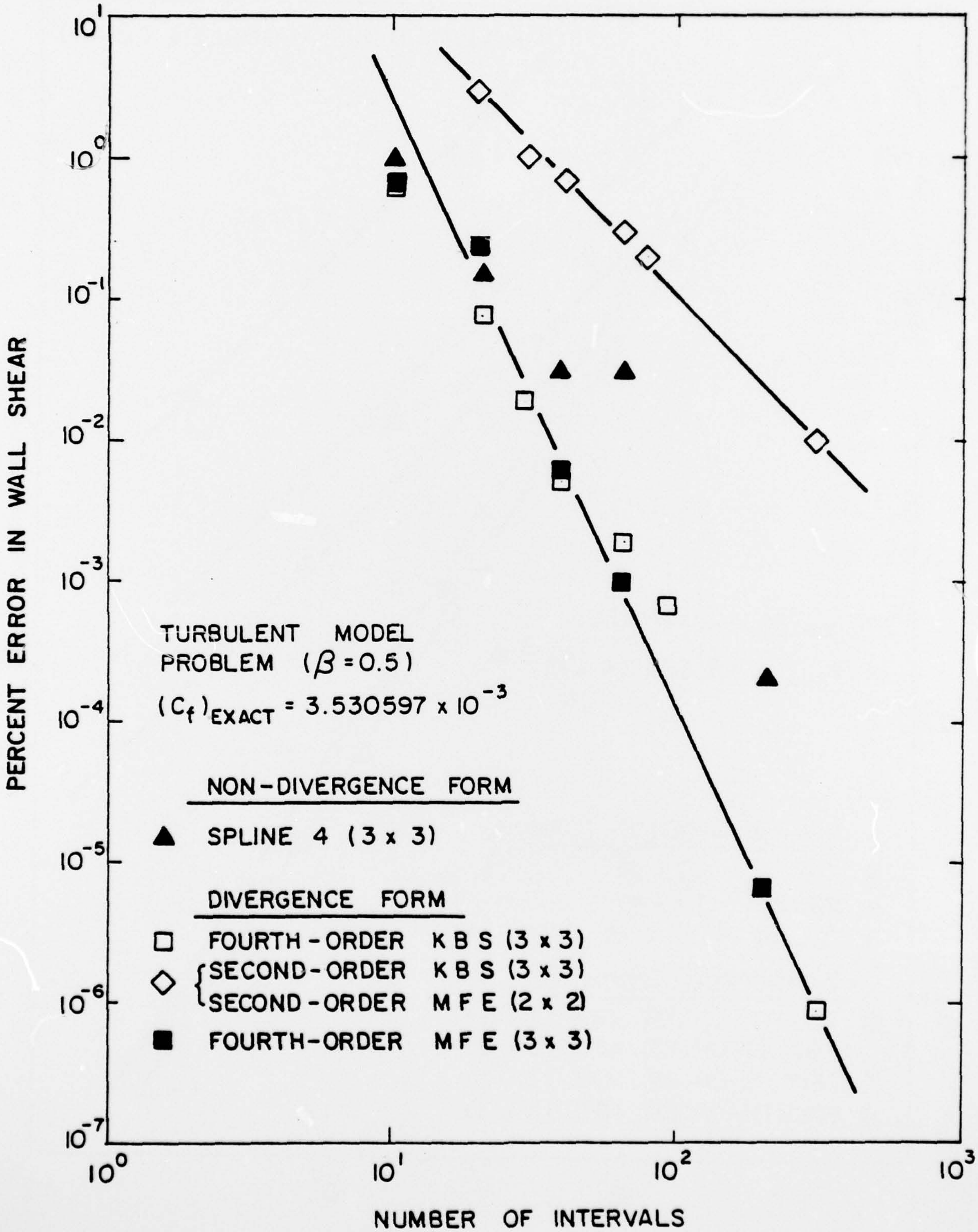

FIG.3 PERCENT ERROR IN THE WALL SHEAR IN TURBULENT MODEL EQUATION (β=0.5)

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REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER AFOSR-TR- 77-1210	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) AN INTEGRAL SPLINE METHOD FOR BOUNDARY LAYER EQUATIONS		5. TYPE OF REPORT & PERIOD COVERED INTERIM 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) S G RUBIN P K KHOSLA		8. CONTRACT OR GRANT NUMBER(s) AFOSR 74-2635
9. PERFORMING ORGANIZATION NAME AND ADDRESS POLYTECHNIC INSTITUTE OF NEW YORK ROUTE 110 FARMINGDALE, NEW YORK 11735		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 2307A1 61102F
11. CONTROLLING OFFICE NAME AND ADDRESS AIR FORCE OFFICE OF SCIENTIFIC RESEARCH/NA BLDG 410 BOLLING AIR FORCE BASE, D C 20332		12. REPORT DATE July 1977 13. NUMBER OF PAGES 41
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

SPLINES
POLYNOMIAL INTERPOLATION
MODIFIED FINITE ELEMENT
BOX METHOD
LAMINAR
TURBULENT

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

An integral procedure using spline polynomials is described for the two-dimensional boundary layer equations. This is a modified finite-element (MFE) formulation, wherein each term in the equations, rather than each independent variable, is approximated with a spline curve fit. Therefore, this is not a true finite-element or Galerkin method and the conventional spline relationships between functional and derivative values still apply. The only difference between the present integral formulation and our earlier differential collocation procedures is in the treatment of the governing differential equations. The

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE UNCLASSIFIED differential methods are more suited to non-conservation equations; the present integral formulation is more desirable for conservation or divergence form of the equations. Boundary layer solutions using conventional second-order finite-difference collocation, the second and fourth order Keller Box Scheme, and fourth-order spline collocation or MFE methods are compared. Conservation and non-conservation forms are considered. Finally, the extension of the MFE formulation to three-coordinate parabolic systems and for the transonic small disturbance equations is briefly described.